

# Non-Euclidean Geometries and the Non-Euclidean World \*

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Axioms in geometry

EVERY CONCLUSION supposes premises; these premises themselves either are self-evident and need no demonstration, or can be established only by relying upon other propositions, and since we can not go back thus to infinity, every deductive science, and in particular geometry, must rest on a certain number of undemonstrable axioms. All treatises on geometry begin, therefore, by the enunciation of these axioms. But among these there is a distinction to be made: Some, for example, 'Things which are equal to the same thing are equal to one another,' are not propositions of geometry, but propositions of analysis. I regard them as analytic judgments *a priori*, and shall not concern myself with them.

But I must lay stress upon other axioms which are peculiar to geometry. Most treatises enunciate three of these explicitly:

- 1° Through two points can pass only one straight;
- 2° The straight line is the shortest path from one point to another;
- 3° Through a given point there is not more than one parallel to a given straight.

Although generally a proof of the second of these axioms is omitted, it would be possible to deduce it from the other two and from those, much more numerous, which are implicitly admitted without enunciating them, as I shall explain further on.

It was long sought in vain to demonstrate likewise the third axiom, known as *Euclid's Postulate*. What vast effort has been wasted in this chimeric hope is truly unimaginable. Finally, in the first quarter of the nineteenth century, and almost at the same time, a Hungarian and a Russian, Bolyai and Lobachevski, established irrefutably that this demonstration is impossible; they have almost rid us of inventors of geometries 'sans postulat'; since then the Académie des Sciences receives only about one or two new demonstrations a year.

The question was not exhausted; it soon made a great stride by the publication of Riemann's celebrated memoir entitled: *Ueber die Hypothesen welche der Geometrie zu Grunde liegen*. This paper has inspired most

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of the recent works of which I shall speak further on, and among which it is proper to cite those of Beltrami and of Helmholtz.

*The Bolyai-Lobachevski Geometry.* If it were possible to deduce Euclid's postulate from the other axioms, it is evident that in denying the postulate and admitting the other axioms, we should be led to contradictory consequences; it would therefore be impossible to base on such premises a coherent geometry.

Now this is precisely what Lobachevski did.

He assumes at the start that: *Through a given point can be drawn two parallels to a given straight.*

And he retains besides all Euclid's other axioms. From these hypotheses he deduces a series of theorems among which it is impossible to find any contradiction, and he constructs a geometry whose faultless logic is inferior in nothing to that of the Euclidean geometry.

The theorems are, of course, very different from those to which we are accustomed, and they can not fail to be at first a little disconcerting.

Thus the sum of the angles of a triangle is always less than two right angles, and the difference between this sum and two right angles is proportional to the surface of the triangle.

It is impossible to construct a figure similar to a given figure but of different dimensions.

If we divide a circumference into  $n$  equal parts, and draw tangents at the points of division, these  $n$  tangents will form a polygon if the radius of the circle is small enough; but if this radius is sufficiently great they will not meet.

It is useless to multiply these examples; Lobachevski's propositions have no relation to those of Euclid, but they are not less logically bound one to another.

*Riemann's Geometry.* Imagine a world uniquely peopled by beings of no thickness (height); and suppose these 'infinitely flat' animals are all in the same plane and can not get out. Admit besides that this world is sufficiently far from others to be free from their influence. While we are making hypotheses, it costs us no more to endow these beings with reason and believe them capable of creating a geometry. In that case, they will certainly attribute to space only two dimensions.

But suppose now that these imaginary animals, while remaining without thickness, have the form of a spherical, and not of a plane figure, and are all on the same sphere without power to get off. What geometry will they construct? First it is clear they will attribute to space only two dimensions; what will play for them the role of the straight line will be the shortest path from one point to another on the sphere, that is to say an arc of a great circle; in a word, their geometry will be the spherical geometry.

What they will call space will be this sphere on which they must stay, and on which happen all the phenomena they can know. Their space will therefore be *unbounded* since on a sphere one can always go forward with-

Main features of Lobachevski geometry

Flatland on a sphere illustrates the ideas of Riemann's geometry (compare Helmholtz)

Attempts to prove the axiom of parallels

out ever being stopped, and yet it will be *finite*; one can never find the end of it, but one can make a tour of it.

Well, Riemann's geometry is spherical geometry extended to three dimensions. To construct it, the German mathematician had to throw overboard, not only Euclid's postulate, but also the first axiom: *Only one straight can pass through two points.*

On a sphere, through two given points we can draw *in general* only one great circle (which, as we have just seen, would play the role of the straight for our imaginary beings); but there is an exception: if the two given points are diametrically opposite, an infinity of great circles can be drawn through them.

In the same way, in Riemann's geometry (at least in one of its forms), through two points will pass in general only a single straight; but there are exceptional cases where through two points an infinity of straights can pass.

*R(iemann) and L(obachevski) contrasted* There is a sort of opposition between Riemann's geometry and that of Lobachevski.

Thus the sum of the angles of a triangle is:

Equal to two right angles in Euclid's geometry;

Less than two right angles in that of Lobachevski;

Greater than two right angles in that of Riemann.

The number of straights through a given point that can be drawn coplanar to a given straight, but nowhere meeting it, is equal:

To one in Euclid's geometry;

To zero in that of Riemann;

To infinity in that of Lobachevski.

Add that Riemann's space is finite, although unbounded in the sense given above to these two words.

*The Surfaces of Constant Curvature.* One objection still remained possible. The theorems of Lobachevski and of Riemann present no contradiction; but however numerous the consequences these two geometers have drawn from their hypotheses, they must have stopped before exhausting them, since their number would be infinite; who can say then that if they had pushed their deductions farther they would not have eventually reached some contradiction?

This difficulty does not exist for Riemann's geometry, provided it is limited to two dimensions; in fact, as we have seen, two-dimensional Riemannian geometry does not differ from spherical geometry, which is only a branch of ordinary geometry, and consequently is beyond all discussion.

Beltrami, in correlating likewise Lobachevski's two-dimensional geometry with a branch of ordinary geometry, has equally refuted the objection so far as it is concerned.

Here is how he accomplished it. Consider any figure on a surface. Imagine this figure traced on a flexible and inextensible canvas applied over this surface in such a way that when the canvas is displaced and

deformed, the various lines of this figure can change their form without changing their length. In general, this flexible and inextensible figure can not be displaced without leaving the surface; but there are certain particular surfaces for which such a movement would be possible; these are the surfaces of constant curvature.

If we resume the comparison made above and imagine beings without thickness living on one of these surfaces, they will regard as possible the motion of a figure all of whose lines remain constant in length. On the contrary, such a movement would appear absurd to animals without thickness living on a surface of variable curvature.

These surfaces of constant curvature are of two sorts: Some are of *positive curvature*, and can be deformed so as to be applied over a sphere. The geometry of these surfaces reduces itself therefore to the spherical geometry, which is that of Riemann.

The others are of *negative curvature*. Beltrami has shown that the geometry of these surfaces is none other than that of Lobachevski. The two-dimensional geometries of Riemann and Lobachevski are thus correlated to the Euclidean geometry.

*Interpretation of Non-Euclidean Geometries.* So vanishes the objection so far as two-dimensional geometries are concerned.

It would be easy to extend Beltrami's reasoning to three-dimensional geometries. The minds that space of four dimensions does not repel will see no difficulty in it, but they are few. I prefer therefore to proceed otherwise.

Consider a certain plane, which I shall call the fundamental plane, and construct a sort of dictionary, by making correspond each to each a double series of terms written in two columns, just as correspond in the ordinary dictionaries the words of two languages whose signification is the same:

*Space:* Portion of space situated above the fundamental plane.

*Plane:* Sphere cutting the fundamental plane orthogonally.

*Straight:* Circle cutting the fundamental plane orthogonally.

*Sphere:* Sphere.

*Circle:* Circle.

*Angle:* Angle.

*Distance between two points:* Logarithm of the cross ratio of these two points and the intersections of the fundamental plane with a circle passing through these two points and cutting it orthogonally.

Etc., Etc.

Now take Lobachevski's theorems and translate them with the aid of this dictionary as we translate a German text with the aid of a German-English dictionary. *We shall thus obtain theorems of the ordinary geometry.* For example, that theorem of Lobachevski: 'the sum of the angles of a triangle is less than two right angles' is translated thus: "If a curvilinear triangle has for sides circle-arcs which prolonged would cut orthogonally the fundamental plane, the sum of the angles of this curvilinear

*Another method for proving consistency will be presented*

*Translation manual for L and E.*

*Are L and R consistent?*

*The consistency of R and L in two dimensions*

*L is consistent,  
because can be  
translated into E*

ear triangle will be less than two right angles." Thus, however far the consequences of Lobachevski's hypotheses are pushed, they will never lead to a contradiction. In fact, if two of Lobachevski's theorems were contradictory, it would be the same with the translations of these two theorems, made by the aid of our dictionary, but these translations are theorems of ordinary geometry and no one doubts that the ordinary geometry is free from contradiction. Whence comes this certainty and is it justified? That is a question I can not treat here because it would require to be enlarged upon, but which is very interesting and I think not insoluble.

*L has  
applications (only  
mathematical  
applications are  
mentioned)*

Nothing remains then of the objection above formulated. This is not all. Lobachevski's geometry, susceptible of a concrete interpretation, ceases to be a vain logical exercise and is capable of applications; I have not the time to speak here of these applications, nor of the aid that Klein and I have gotten from them for the integration of linear differential equations.

This interpretation moreover is not unique, and several dictionaries analogous to the preceding could be constructed, which would enable us by a simple 'translation' to transform Lobachevski's theorems into theorems of ordinary geometry.

*On the Nature of Axioms.* Most mathematicians regard Lobachevski's geometry only as a mere logical curiosity; some of them, however, have gone farther. Since several geometries are possible, is it certain ours is the true one? Experience no doubt teaches us that the sum of the angles of a triangle is equal to two right angles; but this is because the triangles we deal with are too little; the difference, according to Lobachevski, is proportional to the surface of the triangle; will it not perhaps become sensible when we shall operate on larger triangles or when our measurements shall become more precise? The Euclidean geometry would thus be only a provisional geometry.

To discuss this opinion, we should first ask ourselves what is the nature of the geometric axioms.

Are they synthetic *a priori* judgments, as Kant said?

They would then impose themselves upon us with such force, that we could not conceive the contrary proposition, nor build upon it a theoretic edifice. There would be no non-Euclidean geometry.

To be convinced of it take a veritable synthetic *a priori* judgment, the following, for instance:

*If a theorem is true for the number 1, and if it has been proved that it is true of  $n + 1$  provided it is true of  $n$ , it will be true of all the positive whole numbers.*

Then try to escape from that and, denying this proposition, try to found a false arithmetic analogous to non-Euclidean geometry—it can not be done; one would even be tempted at first blush to regard these judgments as analytic.

Moreover, resuming our fiction of animals without thickness, we can hardly admit that these beings, if their minds are like ours, would adopt

*Arithmetic  
is a priori,  
but not  
geometry*

*Arithmetic  
may even  
be analytic!*

the Euclidean geometry which would be contradicted by all their experience.

Should we therefore conclude that the axioms of geometry are experimental verities? But we do not experiment on ideal straight lines or circles; it can only be done on material objects. On what then could be based experiments which should serve as foundation for geometry? The answer is easy.

We have seen above that we constantly reason as if the geometric figures behaved like solids. What geometry would borrow from experience would therefore be the properties of these bodies. The properties of light and its rectilinear propagation have also given rise to some of the propositions of geometry, and in particular those of projective geometry, so that from this point of view one would be tempted to say that metric geometry is the study of solids, and projective, that of light.

But a difficulty remains, and it is insurmountable. If geometry were an experimental science, it would not be an exact science, it would be subject to a continual revision. Nay, it would from this very day be convicted of error, since we know that there is no rigorously rigid solid.

*The axioms of geometry therefore are neither synthetic a priori judgments nor experimental facts.*

They are *conventions*; our choice among all possible conventions is guided by experimental facts; but it remains *free* and is limited only by the necessity of avoiding all contradiction. Thus it is that the postulates can remain *rigorously* true even though the experimental laws which have determined their adoption are only approximative.

In other words, *the axioms of geometry* (I do not speak of those of arithmetic) *are merely disguised definitions.*

Then what are we to think of that question: Is the Euclidean geometry true?

It has no meaning.

As well ask whether the metric system is true and the old measures false; whether Cartesian coordinates are true and polar coordinates false. One geometry can not be more true than another; it can only be *more convenient*.

Now, Euclidean geometry is, and will remain, the most convenient:

1° Because it is the simplest; and it is so not only in consequence of our mental habits, or of I know not what direct intuition that we may have of Euclidean space; it is the simplest in itself, just as a polynomial of the first degree is simpler than one of the second; the formulas of spherical trigonometry are more complicated than those of plane trigonometry, and they would still appear so to an analyst ignorant of their geometric signification.

2° Because it accords sufficiently well with the properties of natural solids, those bodies which our hands and our eye compare and with which we make our instruments of measure.

*If geometry  
were empirical,  
it would not  
have been  
certain*

*Geometry is  
conventional*

*Axioms are  
implicit  
definitions,  
neither true,  
nor false*

*Eternal  
reasons to  
prefer E*



*The Non-Euclidean World.* If geometric space were a frame imposed on *each* of our representations, considered individually, it would be impossible to represent to ourselves an image stripped of this frame, and we could change nothing of our geometry.

But this is not the case; geometry is only the résumé of the laws according to which these images succeed each other. Nothing then prevents us from imagining a series of representations, similar in all points to our ordinary representations but succeeding one another according to laws different from those to which we are accustomed.

We can conceive then that beings who received their education in an environment where these laws were thus upset might have a geometry very different from ours.

Suppose, for example, a world enclosed in a great sphere and subject to the following laws:

The temperature is not uniform; it is greatest at the center, and diminishes in proportion to the distance from the center, to sink to absolute zero when the sphere is reached in which this world is enclosed.

To specify still more precisely the law in accordance with which this temperature varies: Let  $R$  be the radius of the limiting sphere; let  $r$  be the distance of the point considered from the center of this sphere. The absolute temperature shall be proportional to  $R^2 - r^2$ .

I shall further suppose that, in this world, all bodies have the same coefficient of dilatation, so that the length of any rule is proportional to its absolute temperature.

Finally, I shall suppose that a body transported from one point to another of different temperature is put immediately into thermal equilibrium with its new environment.

Nothing in these hypotheses is contradictory or unimaginable.

A movable object will then become smaller and smaller in proportion as it approaches the limit-sphere.

Note first that, though this world is limited from the point of view of our ordinary geometry, it will appear infinite to its inhabitants.

In fact, when these try to approach the limit-sphere, they cool off and become smaller and smaller. Therefore the steps they take are also smaller and smaller, so that they can never reach the limiting sphere.

If, for us, geometry is only the study of the laws according to which rigid solids move, for these imaginary beings it will be the study of the laws of motion of solids *distorted by the differences of temperature* just spoken of.

No doubt, in our world, natural solids likewise undergo variations of form and volume due to warming or cooling. But we neglect these variations in laying the foundations of geometry, because, besides their being very slight, they are irregular and consequently seem to us accidental.

In our hypothetical world, this would no longer be the case, and these variations would follow regular and very simple laws.

Geometry is  
rooted in our  
visual  
perceptions

The heated  
sphere  
(compare  
Reichenbach)

Uniform  
deformations

Moreover, the various solid pieces of which the bodies of its inhabitants would be composed would undergo the same variations of form and volume.

I will make still another hypothesis; I will suppose light traverses media diversely refractive and such that the index of refraction is inversely proportional to  $R^2 - r^2$ . It is easy to see that, under these conditions, the rays of light would not be rectilinear, but circular.

To justify what precedes, it remains for me to show that certain changes in the position of external objects can be *corrected* by correlative movements of the sentient beings inhabiting this imaginary world, and that in such a way as to restore the primitive aggregate of impressions experienced by these sentient beings.

Suppose in fact that an object is displaced, undergoing deformation, not as a rigid solid, but as a solid subjected to unequal dilatations in exact conformity to the law of temperature above supposed. Permit me for brevity to call such a movement a *non-Euclidean displacement*.

If a sentient being happens to be in the neighborhood, his impressions will be modified by the displacement of the object, but he can reestablish them by moving in a suitable manner. It suffices if finally the aggregate of the object and the sentient being, considered as forming a single body, has undergone one of those particular displacements I have just called non-Euclidean. This is possible if it be supposed that the limbs of these beings dilate according to the same law as the other bodies of the world they inhabit.

Although from the point of view of our ordinary geometry there is a deformation of the bodies in this displacement and their various parts are no longer in the same relative position, nevertheless we shall see that the impressions of the sentient being have once more become the same.

In fact, though the mutual distances of the various parts may have varied, yet the parts originally in contact are again in contact. Therefore the tactile impressions have not changed.

On the other hand, taking into account the hypothesis made above in regard to the refraction and the curvature of the rays of light, the visual impressions will also have remained the same.

These imaginary beings will therefore like ourselves be led to classify the phenomena they witness and to distinguish among them the 'changes of position' susceptible of correction by a correlative voluntary movement.

If they construct a geometry, it will not be, as ours is, the study of the movements of our rigid solids; it will be the study of the changes of position which they will thus have distinguished and which are none other than the 'non-Euclidean displacements'; *it will be non-Euclidean geometry*.

Thus beings like ourselves, educated in such a world, would not have the same geometry as ours.

*The World of Four Dimensions.* We can represent to ourselves a four-dimensional world just as well as a non-Euclidean.

Another  
assumption

Bodies and  
measuring  
instruments also  
undergo uniform  
deformations

These  
deformations  
cannot be  
detected from  
within the sphere

Summary

Fourth  
dimension

The sense of sight, even with a single eye, together with the muscular sensations relative to the movements of the eyeball, would suffice to teach us space of three dimensions.

The images of external objects are painted on the retina, which is a two-dimensional canvas; they are *perspectives*.

But, as eye and objects are movable, we see in succession various perspectives of the same body, taken from different points of view.

At the same time, we find that the transition from one perspective to another is often accompanied by muscular sensations.

If the transition from the perspective A to the perspective B, and that from the perspective A' to the perspective B' are accompanied by the same muscular sensations, we liken them one to the other as operations of the same nature.

Studying then the laws according to which these operations combine, we recognize that they form a group, which has the same structure as that of the movements of rigid solids.

Now, we have seen that it is from the properties of this group we have derived the notion of geometric space and that of three dimensions.

We understand thus how the idea of a space of three dimensions could take birth from the pageant of these perspectives, though each of them is of only two dimensions, since *they follow one another according to certain laws*.

Well, just as the perspective of a three-dimensional figure can be made on a plane, we can make that of a four-dimensional figure on a picture of three (or of two) dimensions. To a geometer this is only child's play.

We can even take of the same figure several perspectives from several different points of view.

We can easily represent to ourselves these perspectives, since they are of only three dimensions.

Imagine that the various perspectives of the same object succeed one another, and that the transition from one to the other is accompanied by muscular sensations.

We shall of course consider two of these transitions as two operations of the same nature when they are associated with the same muscular sensations.

Nothing then prevents us from imagining that these operations combine according to any law we choose, for example, so as to form a group with the same structure as that of the movements of a rigid solid of four dimensions.

Here there is nothing unpicturable, and yet these sensations are precisely those which would be felt by a being possessed of a two-dimensional retina who could move in space of four dimensions. In this sense we may say the fourth dimension is imaginable.

**Conclusions.** We see that experience plays an indispensable role in the

genesis of geometry; but it would be an error thence to conclude that geometry is, even in part, an experimental science.

If it were experimental, it would be only approximative and provisional. And what rough approximation!

Geometry would be only the study of the movements of solids; but in reality it is not occupied with natural solids, it has for object certain ideal solids, absolutely rigid, which are only a simplified and very remote image of natural solids.

The notion of these ideal solids is drawn from all parts of our mind, and experience is only an occasion which induces us to bring it forth from them.

The object of geometry is the study of a particular 'group'; but the general group concept pre-exists, at least potentially, in our minds. It is imposed on us, not as form of our sense, but as form of our understanding.

Only, from among all the possible groups, that must be chosen which will be, so to speak, the *standard* to which we shall refer natural phenomena.

Experience guides us in this choice without forcing it upon us; it tells us not which is the truest geometry, but which is the most *convenient*.

Notice that I have been able to describe the fantastic worlds above imagined *without ceasing to employ the language of ordinary geometry*.

And, in fact, we should not have to change it if transported thither.

Beings educated there would doubtless find it more convenient to create a geometry different from ours, and better adapted to their impressions. As for us, in face of the *same* impressions, it is certain we should find it more convenient not to change our habits.

*These concluding remarks will make more sense upon reading Einstein and Reichenbach*

*Geometry is not empirical*

*Geometry studies ideal solids that "pre-exist" in our minds(!)*

*To describe nature, we choose a particular geometry on the basis of convenience*