Intermediate Logic

Lecture notes

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Chapter 7

Modal logic

7.1 Modalities

Modalities (or modals) serve to qualify the truth of a statement:

Roger is $\cdots\cdots$ rich.

Instead of the dots we could put a qualifier such as 'necessarily', 'possibly', 'now', 'then', 'believed by me to be' and so forth. We could also paraphrase that statement to get:

It is $\cdots\cdots$ true that Roger is rich.

So it does not matter whether we qualify the predicate or the whole statement.

We are interested here in necessity and possibility. Many people people believe that things could be different from the way they actually are. Nadal could have won Wimbledon—if Federer had a flu, perhaps. We can isolate the statement about Nadal:

Nadal could win Wimbledon.

We go further and put the modal verb at the front:

It could be that Nadal won Wimbledon.

The possibility indicated by the modal verb 'could' will be represented by a modal operator. A further paraphrase:

Possibly Nadal won Wimbledon. (1)

Now one sense of (1) is epistemic whereby it becomes equivalent to 'For all we know, Nadal won Wimbledon.' This is *not* the meaning we attribute to (1) and synonymous sentences. We take them to mean that there are circumstances alternative to the actual circumstances—for example, the circumstance of Nadal being victorious at Wimbledon.

Necessity and possibility are represented by modal operators $\Box$ and $\Diamond$. Accordingly, to our classical clauses for well-formed formulae we add another one:

FR1. Any of the sentence parameters $P_1, P_2, P_3, \ldots$ is a formula of $\mathsf{H_s}$;

FR2. If A is a formula, then so is $\neg A$;

FR3. If A and B are formulae, then so is $(A \supset B)$;

FR4. If A is a formula, then so are $\Box A$ and $\Diamond A$.

7.2 Modal square

What is the relation between necessity and possibility? Aristotle worked out the basic connections.

7.3 Semantics

Consider again our modal modifier:

Roger is $\cdots\cdots$ is rich.

We can put the modifier at the end:

$$\text{Roger is rich } \cdots\cdots. \tag{1}$$

We can the modifier to refer to a particular circumstance. Hence, if Roger is rich necessarily, he is so in *every* circumstance. If he is rich possibly, he is so in *some* circumstance. In the classical logic we have already seen that circumstances are fixed by valuations. Therefore, (1) is paraphrased as:

Roger is rich under valuation i.

Necessity and possibility will accordingly take the form:

For all i, Roger is rich in i.
For some i, Roger is rich in i.

Example 7.1. Consider the formula $\Box P \supset P$. It says that if P is true under every valuation, it is true under the actual valuation. So this is an intuitively valid claim. By contrast, consider $P \supset \Box P$. Merely because P is true in the actual circumstance, it is not true in every other circumstance. So this claim is invalid.

Possible valuations are commonly known as 'possible worlds'. Modal semantics uses another, not quite so intuitive notion of *accessibility* between worlds. Let us illustrate it with following example.

Example 7.2 (Accessibility)**.** Let $\Diamond P$ be interpreted as 'It is conceivable that P' and consider the formula:

$$\Diamond P \supset \Box\Diamond P. \tag{2}$$

The formula (2) says that if P is true in some conceivable world—the one we can conceive from the actual world—then from every conceivable world there is a conceivable world in which P is true. Now from the perspective of a medieval monk it is inconceivable that the Earth is four billion years old. But it is conceivable to us. So the antecedent of (2) is true, but the consequent is false.

Let us proceed with a formal treatment of semantics.

Definition 7.3. A *frame* is a pair $\langle \mathscr{W}, \mathscr{R} \rangle$ where $\mathscr{W}$ is a set of possible worlds and $\mathscr{R}$ is the accessibility relation defined on $\mathscr{W}$.

Definition 7.4. A *propositional modal model* is a triple $\langle \mathscr{W}, \mathscr{R}, \Vdash \rangle$, where $\Vdash$ is a relation between possible worlds and sentential parameters. In case $\Gamma \Vdash P$ holds, we say that P is true on Γ, otherwise we say it is false on Γ.

Definition 7.5. Let $\langle \mathscr{W}, \mathscr{R}, \Vdash \rangle$ be a model. Then for each $\Gamma \in \mathscr{W}$:

1. $\Gamma \Vdash \neg A$ iff $\Gamma \nVdash A$
2. $\Gamma \Vdash A \supset B$ iff $\Gamma \nVdash A$ or $\Gamma \Vdash B$
3. $\Gamma \Vdash \Box A$ iff for every $\Delta \in \mathscr{W}$, if $\Gamma \mathscr{R} \Delta$, then $\Delta \Vdash A$

To illustrate the significance of the accessibility relation and the notion of truth in a modal model, we shall provide diagrams.

Example 7.6. See Figure 7.1. Here we have that, since $\Delta \Vdash P$, $\Delta \Vdash P \vee Q$. Similarly $\Omega \Vdash P \vee Q$. Now $\Gamma \Vdash \Box(P \vee Q)$. But neither $\Gamma \Vdash \Box P$, nor $\Gamma \Vdash \Box Q$. So $\Gamma \nVdash (\Box P \vee \Box Q)$. Therefore, $\Gamma \nVdash (\Box(P \vee Q) \supset (\Box P \vee \Box Q))$.

Example 7.7. See Figure 7.2. Here we have that $\Gamma \nVdash (\Diamond P \supset \Box\Diamond P)$.

We now turn to the issue of showing that some formulae are true at the worlds. The diagrams do not help as they provide *counter-examples*.

Proposition 7.8. *Let $\langle \mathscr{W}, \mathscr{R}, \Vdash \rangle$ be a model, and let $\Gamma \in \mathscr{W}$. Then, if $\Gamma \Vdash \Box(P \wedge Q)$, then $\Gamma \Vdash (\Box P \wedge \Box Q)$.*

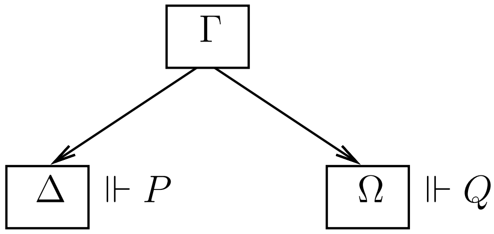

Figure 7.1: No distribution of necessity

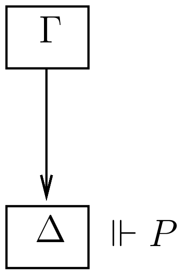

Figure 7.2: No necessitation of possibility

Proof. Working straight from definitions:

$$
\begin{array}{ll}
\Gamma \Vdash \Box(P \wedge Q) & \\
\Delta \in \mathscr{W} : \Gamma \mathscr{R} \Delta & \Delta \text{ is arbitrary} \\
\Delta \Vdash (P \wedge Q) & \\
\Delta \Vdash P & \\
\Delta \Vdash Q & \\
\Gamma \Vdash \Box P & \text{since } \Delta \text{ is arbitrary} \\
\Gamma \Vdash \Box Q & \text{since } \Delta \text{ is arbitrary} \\
\Gamma \Vdash (\Box P \wedge \Box Q). &
\end{array}
$$

□

Proposition 7.9. *If $\mathscr{R}$ is reflexive, then, for every $\Gamma \in \mathscr{W}$, $\Gamma \Vdash (\Box P \supset P)$.*

Proof. We prove by *reductio*:

$$
\begin{array}{ll}
\Gamma \Vdash \Box P & \\
\Gamma \mathscr{R} \Gamma & \text{reflexivity} \\
\Gamma \Vdash P & \\
\Gamma \nVdash P & \text{For } \textit{reductio} \\
\bot. &
\end{array}
$$

□

Proposition 7.10. *If $\mathscr{R}$ is symmetric, then, for every $\Gamma \in \mathscr{W}$, $\Gamma \Vdash (P \supset \Box\Diamond P)$.*

Proof. Again by *reductio*:

$\Gamma \Vdash P$	
$\Gamma \nVdash \Box\Diamond P$	For *reductio*
$\Gamma\mathscr{R}\Delta : \Delta \nVdash \Diamond P$	
$\forall w \in \mathscr{W} : \Delta\mathscr{R}w \Rightarrow w \nVdash P$	
$\Delta\mathscr{R}\Gamma$	symmetry
$\Gamma \nVdash P$	
$\bot$.	

$\Box$

Proposition 7.11. *If $\mathscr{R}$ is symmetric and transitive, then, for every $\Gamma \in \mathscr{W}$, $\Gamma \Vdash (\Diamond P \supset \Box\Diamond P)$.*

Proof. Again by *reductio*:

$\Gamma \Vdash \Diamond P$	
$\Gamma\mathscr{R}\Omega : \Omega \Vdash P$	
$\Gamma \nVdash \Box\Diamond P$	For *reductio*
$\Gamma\mathscr{R}\Delta : \Delta \nVdash \Diamond P$	
$\forall w \in \mathscr{W} : \Delta\mathscr{R}w \Rightarrow w \nVdash P$	
$\Delta\mathscr{R}\Gamma$	symmetry
$\Delta\mathscr{R}\Omega$	transitivity
$\Omega \nVdash P$	
$\bot$.	

$\Box$

7.4 Tableau rules

We modify familiar tableau rules to fit the semantics for modal logic.

Definition 7.12. A *modal prefix* is a finite sequence of positive integers. A prefixed formula is an expression of the form $\xi\, A$, where ξ is a modal prefix and A is a signed formula.

Example 7.13. '1.2.4.7' is a prefix. Moreover, if ξ is a prefix and n is a positive integer, then $\xi.n$ is also a prefix.

Intuitively the prefix designates a possible world, and $\xi\, A$ says that A is true on ξ. The prefix $\xi.n$ designates a possible world accessible from ξ. The modified tableau rules are given in Table 7.4. All the definitions and applications of the tableau method carry over from the classical case, with the following amendment:

Definition 7.14. A *conjugate* pair of formulae is a pair $\langle \xi\, \mathbf{T}A, \xi\, \mathbf{F}A \rangle$.

Our rules correspond to the basic modal system **K** in which no restrictions are imposed on relation $\mathscr{R}$. It is a very weak system in which many intuitive properties of necessity and possibility cannot be proven.

Remark. We have to exercise care in applying the rule for necessity. The rule says that we can proceed from $\xi\, \mathbf{T}\Box A$ to $\xi.n\, \mathbf{T}A$ *provided* there is a world designated by $\xi.n$—that is, there is a world accessible from ξ. The difficulty here corresponds to the situation in the predicate logic when we work with an empty domain. It can be illustrated by the tableau in Figure 7.3 where we attempt to prove the formula $\Box A \supset \Diamond A$. Accordingly, we in effect should make the proviso that the necessity-like rules can employ the prefix $\xi.n$ if it already occurs in the tableau above, thus limiting the analogy with the universal quantifier rules.

Example 7.15. Show that $\vDash (\Box P \wedge \Box Q) \supset \Box(P \wedge Q)$. The desired tableau is in Figure 7.4.

The properties of necessity and possibility may be explored either syntactically or semantically. The syntactic way means having additional tableau rules governing the operators of necessity and possibility. The semantic way means imposing restrictions on the accessibility relation in the modal model. We have seen

$$
\begin{array}{c}
\vdots \\
\xi\,\mathbf{T}\neg A \\
\vdots \\
\\
\xi\,\mathbf{F}A
\end{array}
\qquad
\begin{array}{c}
\vdots \\
\xi\,\mathbf{F}\neg A \\
\vdots \\
\\
\xi\,\mathbf{T}A
\end{array}
\qquad
\begin{array}{cc}
\multicolumn{2}{c}{\vdots} \\
\multicolumn{2}{c}{\xi\,\mathbf{T}A \supset B} \\
\multicolumn{2}{c}{\vdots} \\
\\
\xi\,\mathbf{F}A & \xi\,\mathbf{T}B
\end{array}
\qquad
\begin{array}{c}
\vdots \\
\xi\,\mathbf{F}A \supset B \\
\vdots \\
\\
\xi\,\mathbf{T}A \\
\xi\,\mathbf{F}B
\end{array}
$$

$$
\begin{array}{c}
\vdots \\
\xi\,\mathbf{T}A \wedge B \\
\vdots \\
\\
\xi\,\mathbf{T}A \\
\xi\,\mathbf{T}B
\end{array}
\qquad
\begin{array}{cc}
\multicolumn{2}{c}{\vdots} \\
\multicolumn{2}{c}{\xi\,\mathbf{F}A \wedge B} \\
\multicolumn{2}{c}{\vdots} \\
\\
\xi\,\mathbf{F}A & \xi\,\mathbf{F}B
\end{array}
\qquad
\begin{array}{cc}
\multicolumn{2}{c}{\vdots} \\
\multicolumn{2}{c}{\xi\,\mathbf{T}A \vee B} \\
\multicolumn{2}{c}{\vdots} \\
\\
\xi\,\mathbf{T}A & \xi\,\mathbf{T}B
\end{array}
\qquad
\begin{array}{c}
\vdots \\
\xi\,\mathbf{F}A \wedge B \\
\vdots \\
\\
\xi\,\mathbf{F}A \\
\xi\,\mathbf{F}B
\end{array}
$$

$$
\begin{array}{cc}
\multicolumn{2}{c}{\vdots} \\
\multicolumn{2}{c}{\xi\,\mathbf{T}A \leftrightarrow B} \\
\multicolumn{2}{c}{\vdots} \\
\\
\xi\,\mathbf{T}A & \xi\,\mathbf{F}A \\
\xi\,\mathbf{T}B & \xi\,\mathbf{F}B
\end{array}
\qquad
\begin{array}{cc}
\multicolumn{2}{c}{\vdots} \\
\multicolumn{2}{c}{\xi\,\mathbf{F}A \leftrightarrow B} \\
\multicolumn{2}{c}{\vdots} \\
\\
\xi\,\mathbf{T}A & \xi\,\mathbf{F}A \\
\xi\,\mathbf{F}B & \xi\,\mathbf{T}B
\end{array}
$$

$$
\begin{array}{c}
\vdots \\
\xi\,\mathbf{T}\Diamond A \\
\vdots \\
\\
\xi.n\,\mathbf{T}A \\
(\xi.n\ \text{ is `new'})
\end{array}
\qquad
\begin{array}{c}
\vdots \\
\xi\,\mathbf{F}\Box A \\
\vdots \\
\\
\xi.n\,\mathbf{F}A \\
(\xi.n\ \text{ is `new'})
\end{array}
\qquad
\begin{array}{c}
\vdots \\
\xi\,\mathbf{T}\Box A \\
\vdots \\
\\
\xi.n\,\mathbf{T}A \\
(\xi.n\ \text{ is arbitrary})
\end{array}
\qquad
\begin{array}{c}
\vdots \\
\xi\,\mathbf{F}\Diamond A \\
\vdots \\
\\
\xi.n\,\mathbf{F}A \\
(\xi.n\ \text{ is arbitrary})
\end{array}
$$

Table 7.1: Signed modal tableau processing rules

$1\mathbf{T}\Box A$
$1\mathbf{F}\Diamond A$

~~$1.1\mathbf{T}A$~~

~~$1.1\mathbf{F}A$~~

Figure 7.3: $\Box A \supset \Diamond A$ is not **K**-valid.

$1\mathbf{T}\Box P \wedge \Box Q$
$1\mathbf{F}\Box(P \wedge Q)$

$1\mathbf{T}\Box P$
$1\mathbf{T}\Box Q$

$1.1\mathbf{F}P \wedge Q$

$1.1\mathbf{F}P$ $\quad$ $1.1\mathbf{F}Q$

$1.1\mathbf{T}P$ $\quad$ $1.1\mathbf{T}Q$
$\times$ $\quad$ $\times$

Figure 7.4: $(\Box P \wedge \Box Q) \supset \Box(P \wedge Q)$ is **K**-valid.

above in Propositions 7.9, 7.10, and 7.11 that the two ways are correlated with each other (at least in some cases). The special tableau rules are given in Table 7.4. The relevant logics obtain as follows:

D	D	serial
T	T	reflexive
K4	4	transitive
B	$B, 4$	reflexive, symmetric
S4	$T, 4$	reflexive, transitive
S5	$T, 4, 4r$	equivalence.

Example 7.16. Let us show that $\Diamond(A \wedge \Box B) \leftrightarrow (\Diamond A \wedge \Box B)$ is **S5**-valid. The relevant tableau is in Figure 7.5.

T	$\begin{array}{c}\vdots\\ \xi\,\mathbf{T}\Box A\\ \vdots\\ \\ \xi\,\mathbf{T}A\end{array}$	$\begin{array}{c}\vdots\\ \xi\,\mathbf{F}\Diamond A\\ \vdots\\ \\ \xi\,\mathbf{F}A\end{array}$
D	$\begin{array}{c}\vdots\\ \xi\,\mathbf{T}\Box A\\ \vdots\\ \\ \xi\,\mathbf{T}\Diamond A\end{array}$	$\begin{array}{c}\vdots\\ \xi\,\mathbf{F}\Diamond A\\ \vdots\\ \\ \xi\,\mathbf{F}\Box A\end{array}$
B	$\begin{array}{c}\vdots\\ \xi.n\,\mathbf{T}\Box A\\ \vdots\\ \\ \xi\,\mathbf{T}A\end{array}$	$\begin{array}{c}\vdots\\ \xi.n\,\mathbf{F}\Diamond A\\ \vdots\\ \\ \xi\,\mathbf{F}A\end{array}$
4	$\begin{array}{c}\vdots\\ \xi\,\mathbf{T}\Box A\\ \vdots\\ \\ \xi.n\,\mathbf{T}\Box A\end{array}$	$\begin{array}{c}\vdots\\ \xi\,\mathbf{F}\Diamond A\\ \vdots\\ \\ \xi.n\,\mathbf{F}\Diamond A\end{array}$
4r	$\begin{array}{c}\vdots\\ \xi.n\,\mathbf{T}\Box A\\ \vdots\\ \\ \xi\,\mathbf{T}\Box A\end{array}$	$\begin{array}{c}\vdots\\ \xi.n\,\mathbf{F}\Diamond A\\ \vdots\\ \\ \xi\,\mathbf{F}\Diamond A\end{array}$

Table 7.2: Additional tableau processing rules

7.5 Digression: axioms

Let us first list the additional axioms for some particularly important logics:

K	$\Box(A \supset B) \supset (\Box A \supset \Box B)$
D	$\Box A \supset \Diamond A$
T	$\Box A \supset A$
4	$\Box A \supset \Box\Box A$
B	$A \supset \Box\Diamond A$
5	$\Diamond A \supset \Box\Diamond A$.

The relevant logics obtain as follows:

K	K
D	D
T	T
K4	4
B	T, B
S4	$T, 4$
S5	$T, 5$ or $T, 4, B$.

$$1\mathbf{F}\Diamond(A \wedge \Box B) \leftrightarrow (\Diamond A \wedge \Box B)$$

$$1\mathbf{T}\Diamond(A \wedge \Box B) \qquad 1\mathbf{F}\Diamond(A \wedge \Box B)$$
$$1\mathbf{F}(\Diamond A \wedge \Box B) \qquad 1\mathbf{T}(\Diamond A \wedge \Box B)$$

$$1.1\mathbf{T}A \wedge \Box B \qquad 1\mathbf{T}\Diamond A$$
$$\qquad 1\mathbf{T}\Box B$$

$$1.1\mathbf{T}A \qquad 1.1\mathbf{T}A$$
$$1.1\mathbf{T}\Box B$$

$$1\mathbf{T}B \qquad 1.1\mathbf{F}A \wedge \Box B$$

$$1\mathbf{F}\Diamond A \qquad 1\mathbf{F}\Box B \qquad 1.1\mathbf{F}A \qquad 1.1\mathbf{F}\Box B$$
$$\times \qquad \times$$

$$1.1\mathbf{F}A \qquad 1\mathbf{F}\Box B$$
$$\times \qquad \times$$

Figure 7.5: $\Diamond(A \wedge \Box B) \leftrightarrow (\Diamond A \wedge \Box B)$ is **S5**-valid.